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OPTIMUM GROWTH AND ALLOCATION OF FOREIGN EXCHANGE

by

Pranab K. Bardhan

No. 36

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**massachusetts  
institute of  
technology**

**50 memorial drive  
cambridge, mass. 02139**

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The views expressed in this paper are the author's sole responsibility, and do not reflect those of the Department of Economic, nor of the Massachusetts Institute of Technology

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Massachusetts Institute of Technology

## I INTRODUCTION

In this paper we try to analyze the intertemporally optimal pattern of allocation of foreign exchange in a simple growth model with three sectors. One sector is that producing consumption goods (say, food grains), one producing capital goods (say, tractors) directly for the consumer-goods sector and the third representing capital goods (say, machine tools) required for further production of capital goods; the latter two sectors may be regarded as sub-branches of the "department" of capital goods in the original Marxian scheme of reproduction. <sup>1/</sup> In the literature on development planning a question often arises about the best way of allocating a given amount of foreign exchange (received as foreign aid or earned through exports) among these three sectors so as to optimize the social objective function over time. <sup>2/</sup> The question was once vaguely hinted (but not correctly posed or answered) by Mohalanobis [11] in connection with Indian planning. It was made more explicit by Raj and Sen [13] who considered a four-sector model (with an added intermediate-goods

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\* Frequent and fairly long conversations on this paper with Sanjit Bose were a substantial help. At various stages of writing this paper I have also benefited from discussions with Anthony Atkinson, James Mirrlees, Harl Ryder, Robert Solow and T.N. Srinivasan. All errors, needless to say, are mine alone.

1. This sub-classification of the Marxian "departments" and particularly the strategic role of the machine-tools sector in development planning have been emphasized by Maurice Dobb and others. See Dobb [6], Ch. IV, and Raj [12].
2. This type of question regarding the optimum allocation of resources (investment and foreign exchange) to the capital goods sector and the machine-tools sector in particular, has played an important role in the discussion of development planning. On the question of proper allocation to the machine-tools sector it has sometimes been pointed out, for example, that compared to the Indian planners, Chinese planners have allocated a far larger proportion of investment and foreign exchange to machine-building, and from the long-run point of view this is sometimes regarded as a better policy, see [12]. Our paper is meant to provide a simple theoretical framework for discussing this kind of issue.

sector)<sup>3/</sup> and calculated the rates of growth of consumption under four alternative ways of spending the given amount of foreign exchange. Their essential purpose was only to show the conflict between short-run and long-run benefits of any particular policy of exchange allocation, a conflict that is at the heart of much of capital theory. As Solow [15] once pointed out in a review of Dobb's book [6]:

"The existence of a machine-tool sector opens up the possibility of postponing an increase in consumption and in the investable surplus, and concentrating instead on producing more and more efficient machine tools so that, at a later date, large-scale production of tractors and of consumables can be begun. (I am told that there has been discussion in India of whether to import fertilizer, or import fertilizer-making machines, or import machinery for making fertilizer-making machinery.) This is the kind of problem that can be solved only by Ramsey-Fisher considerations and not by any simple growth-rate comparison."

This paper is an attempt at the type of Ramsey-Fisher exercise that Solow seems to have in mind.

The problem of optimal allocation of exchange (or any other resource) becomes particularly important in our sectoral growth model because the latter differs from the usual neoclassical growth models in assuming "non-shiftability" of capital, i.e., capital, once installed in a sector, cannot be used in any other. It is this irreversibility of investment that is at the root of different policies of sectoral allocation of current resources giving rise to different time-profiles of consumption. In this respect our three-sector production model belongs to the same class as that of the two-sector growth models associated with Fel'dman [8], Domar [7] and Mohalanobis [10]. As in these models we also abstract from labour as a primary factor of production.

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3. In this paper we avoid the extra complications that follow from the introduction of this fourth sector.

Very recently, optimal investment allocation in such a two-sector growth model has been analyzed by Johansen [9], Chakravarty [4] and more completely, by Bose [2] and Dasgupta [5]. Our paper is in the same spirit, but it introduces the machine-tool sector and is oriented to allocation of foreign exchange, which may be given exogenously to the system (as foreign aid), or may be earned through exports of consumer goods which reduce current consumption. In the model of Section III where there is domestic production of machine tools, we analyze, apart from the optimum sectoral allocation of foreign exchange, the optimum allocation of the currently available flow of machine tools between using them for producing tractors and that for producing more machine tools.

Our aim is to maximize over a finite horizon  $T$ , the following social objective function:

$$\int_0^T e^{-\rho t} C(t) dt + K_c(T) e^{-\rho T} \quad (1)$$

where  $C$  is consumption,  $K_c(T)$  is terminal capacity in the consumer goods sector, and  $\rho$  is the given social rate of time discount. We are aware of the numerous objections one may have about the appropriateness of this objective function. But, as the reader should have guessed, some of the simplifications are assumed to keep the problem analytically tractable. In the Appendix to this paper we give some results for the case when we have an infinite horizon and a non-linear instantaneous utility function. It might have been better possibly if a "scrap-value" function involving all types of capital were included in the maximand along with the utility functional, but in view of the relatively complicated structure of capital in our model we find it more convenient to include only the terminal capacity in the consumer goods sector (see also footnote 5).



## II

### OPTIMUM ALLOCATION OF FOREIGN EXCHANGE WITH NO DOMESTIC PRODUCTION OF MACHINE TOOLS

We are abstracting from labour as a factor of production, as we have already mentioned, and we keep a constant capital-output ratio in each sector.  $\beta_i$  is the output-capital ratio in the  $i^{\text{th}}$  sector,  $K_i$  is the capital stock in the  $i^{\text{th}}$  sector and  $\alpha_i$  is the proportion of foreign exchange spent in importing the  $i^{\text{th}}$  type good. Let us start with the case where we get a constant amount of foreign exchange,  $F$ , each year through foreign aid (gift); we shall later take the case of foreign exchange earned through exports. We assume that ours is a small country which takes the world prices as given (assumed to be unity, without further loss of generality).

Our consumption is thus given by:

$$C(t) = \beta_c K_c(t) + \alpha_c(t)F \quad (2)$$

The rate of accumulation of  $K_c$  ("tractors") is given by:

$$\dot{K}_c(t) = \beta_I K_I(t) + \alpha_I(t)F \quad (3)$$

For the time being let us assume that there is no domestic production of  $K_I$ , "machine tools" which produce "tractors," and import is the only source of getting them. (In Section III we shall introduce machine tools production at home.) Thus

$$\dot{K}_I(t) = \alpha_m(t)F \quad (4)$$

---

4. We are ignoring depreciation of all types of capital. A proportional rate of depreciation is not difficult to introduce in this model.

Since the  $\alpha_i$ 's are proportions,

$$\alpha_c + \alpha_I + \alpha_m = 1 \quad (5)$$

Initial stocks of capital,  $K_c(0)$  and  $K_I(0)$  are given. The Hamiltonian  $H$  of the present problem is given by

$$\begin{aligned} He^{\rho t} = & [\beta_c K(t) + \alpha_c(t)F] + q_c(t)[\beta_I K_I(t) + \alpha_I(t)F] \\ & + q_I(t)[1 - \alpha_c(t) - \alpha_I(t)]F \end{aligned} \quad (6)$$

where  $q_c$  and  $q_I$  are the imputed prices of investment,  $\dot{K}_c$  and  $\dot{K}_I$ , respectively.

The optimality conditions are given as follows:

$$\alpha_I(t) \begin{cases} = 0 & \text{if } q_c(t) < q_I(t) \\ \in [0, 1 - \alpha_c(t)] & \text{if } q_c(t) = q_I(t) \\ = 1 - \alpha_c(t) & \text{if } q_c(t) > q_I(t) \end{cases} \quad (7)$$

$$\alpha_c(t) \begin{cases} = 0 & \text{if } \max[q_c(t), q_I(t)] > 1 \\ \in [0, 1] & \text{if } \max[q_c(t), q_I(t)] = 1 \\ = 1 & \text{if } \max[q_c(t), q_I(t)] < 1 \end{cases} \quad (8)$$

$q_i(t)$  are continuous satisfying the following:

$$\dot{q}_c(t) = \rho q_c(t) - \beta_c \quad (9)$$

$$\dot{q}_I(t) = \rho q_I(t) - \beta_I q_c(t) \quad (10)$$

$$q_c(T) = 1 \quad (11)$$

$$\text{and } q_I(T) = 0 \quad (12)$$

Now from (9) - (12)

$$q_c(t) = \frac{\beta_c}{\rho} - e^{-\rho(T-t)} \frac{(\beta_c - \rho)}{\rho} \quad (13)$$

$$q_I(t) = \frac{\beta_I}{\rho} \left[ \frac{\beta_c}{\rho} \{1 - e^{-\rho(T-t)}\} - (\beta_c - \rho)(T-t)e^{-\rho(T-t)} \right] \quad (14)$$

In this paper we shall assume that  $\beta_I > \rho$ , i.e., society is not too impatient to exploit the productivity of capital in any sector.

From (11) and (13) we can immediately see that  $q_c(t)$  is always greater than unity for  $t < T$ , so that  $\max[q_c(t), q_I(t)] > 1$ . From (8), this means  $\alpha_c = 0$ , i.e., on the optimum path we do not spend any part of foreign exchange on consumer goods. <sup>5/</sup>

From (13) and (14),

$$q_I(T-t) - q_c(T-t) = \frac{e^{-\rho t}}{\rho^2} [(e^{\rho t} - 1)\beta_c(\beta_I - \rho) - \beta_I(\beta_c - \rho)\rho t - \rho^2] \quad (15)$$

If we call the bracketed expression on the R.H.S. of (15)  $f(x)$ , where  $x = \rho t$ , then it is easy to see that there exists a unique positive  $x^*$  such that  $f(x^*) = 0$ . This is shown in Figure 1. It is also easy to establish that  $f'(x^*) > 0$ . All this means that there exists a unique positive  $\tau$  such that

$$q_I(T-t) - q_c(T-t) \begin{cases} > 0 \\ < 0 \end{cases} \text{ as } t \begin{cases} > \\ < \end{cases} \tau. \quad (16)$$

Figure 2 depicts this.

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5. This result will change if we do not include  $K_c(T)$  in the objective function along with the utility functional. If no terminal capacity constraints are assumed, then  $q_c(T) = q_I(T) = 0$ . In this case we shall have a terminal phase on the optimum path when we import consumer goods. On the other hand if we have a scrap value function which is  $[K_c(T) + K_I(T)]e^{-\rho T}$ , then once again on the optimum path  $\alpha_c = 0$ ; but in this case the sign of (15) would depend on a comparison of  $\beta_I$  and  $\beta_c$ .

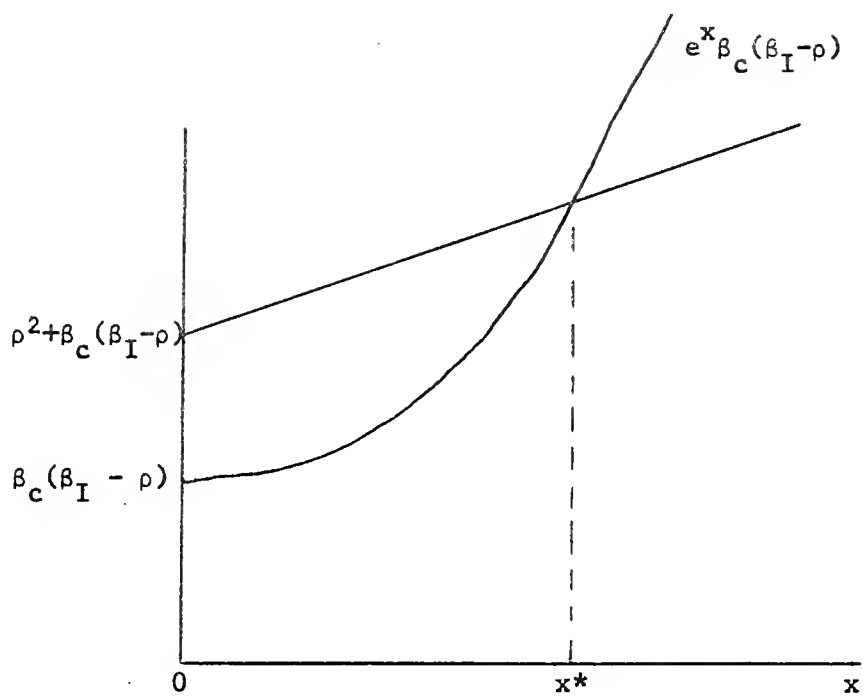


FIGURE 1

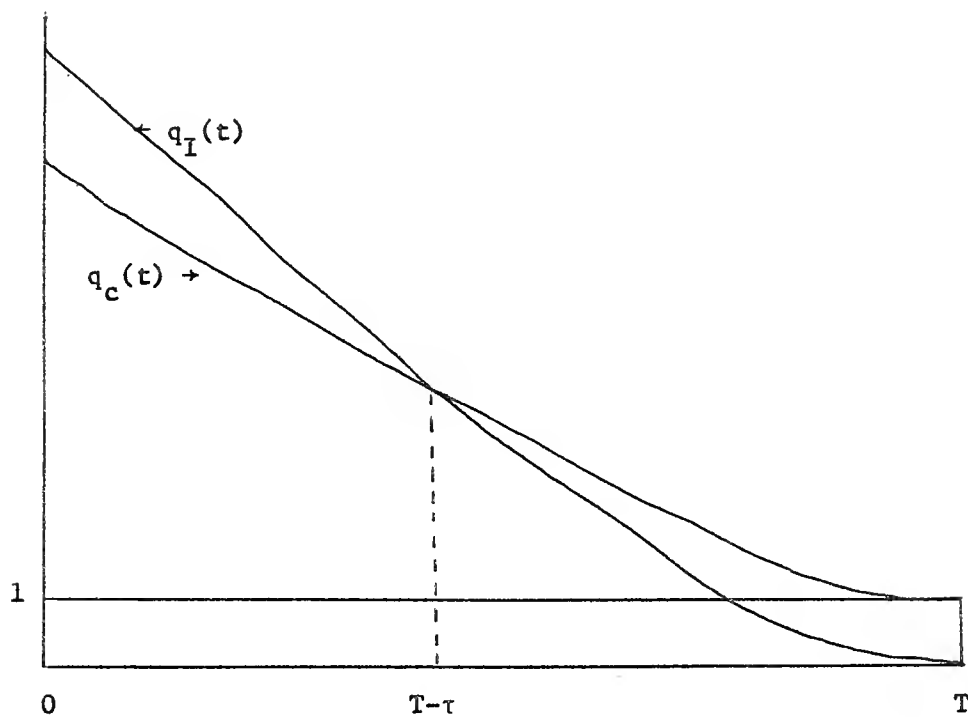


FIGURE 2

From (7) we may now say that on the optimum path, in the first phase (up to time  $T-\tau$ ) we spend all of our foreign exchange  $F$  in importing "machine tools" ( $\alpha_m=1, \alpha_I=\alpha_c=0$ ), and in the second phase we spend all of it in importing "tractors" ( $\alpha_I=1, \alpha_m=\alpha_c=0$ ). It is also important to note that  $\tau$  is independent of  $T$ , so that the time spent on the optimum path when all foreign exchange is used to buy "tractors" is the same irrespective of the planning horizon and the longer is the horizon the larger is the proportion of time on the optimum path when all foreign exchange is spent on importing machine tools.

In the Appendix to this paper we consider the same allocation problem but with a planning horizon that is infinite and with a non-linear instantaneous utility function. We have not been able to completely characterize the optimum path but on the basis of our calculations in the Appendix, we can report here that  $\alpha_c(t) = \alpha_I(t) = 0$  and  $\alpha_m(t) = 1$  is the asymptotically optimal policy, i.e., "in the long run" we should spend all of our foreign exchange in importing "machine tools." This ties up with our finding in the finite-horizon case that the longer is the planning horizon the larger is the proportion of time spent in importing only machine tools.

### III

#### OPTIMUM ALLOCATION WITH DOMESTIC PRODUCTION OF MACHINE TOOLS

In this section we go back to the finite-horizon case but make the model more complicated by introducing domestic production of machine tools. These machine tools may be used to produce either tractors or more machine tools, and the allocation of the machine tools between the two uses has now to be decided, apart from the sectoral allocation of foreign exchange, on the basis of the optimization process.

Equation (4) in Section II has now to be rewritten as

$$\dot{K}_I(t) + \dot{K}_m(t) = \beta_m K_m(t) + \alpha_m(t)F \quad (17)$$

$$\dot{K}_I = \lambda(t)[\dot{K}_I + \dot{K}_m] \quad (18)$$

where  $\lambda$  is the proportion of the amount of machine tools currently available to be used in the production of tractors.

$$1 \geq \lambda(t) \geq 0 \quad (19)$$

The Hamiltonian  $H$  of (6) has now to be rewritten as

$$\begin{aligned} He^{pt} = & [\beta_c K_c(t) + \alpha_c(t)F] + q_c(t)[\beta_I K_I(t) + \alpha_I(t)F] \\ & + q(t)[\beta_m K_m(t) + \{1 - \alpha_c(t) - \alpha_I(t)\}F] \end{aligned} \quad (20)$$

where  $q(t) = \lambda(t)q_I(t) + [1 - \lambda(t)]q_m(t)$  and  $q_m(t)$  is the imputed price of investment in the machine tools sector.

The optimality conditions are as follows:

$$\lambda(t) \begin{cases} = 0 & \text{if } q_I(t) < q_m(t) \\ \in [0, 1] & \text{if } q_I(t) = q_m(t) \\ = 1 & \text{if } q_I(t) > q_m(t) \end{cases} \quad (21)$$

$$\text{so that } q(t) = \max[q_I(t), q_m(t)] \quad (22)$$

$$\alpha_I(t) \begin{cases} = 0 & \text{if } q_c(t) < q(t) \\ \in [0, 1 - \alpha_c(t)] & \text{if } q_c(t) = q(t) \\ = 1 - \alpha_c(t) & \text{if } q_c(t) > q(t) \end{cases} \quad (23)$$

$$\alpha_c(t) \begin{cases} = 0 & \text{if } \max[q_c(t), q(t)] > 1 \\ \in [0, 1] & \text{if } \max[q_c(t), q(t)] = 1 \\ = 1 & \text{if } \max[q_c(t), q(t)] < 1, \end{cases} \quad (24)$$

$q_I(t)$  are continuous satisfying (9), (10), (11) and

$$\dot{q}_m(t) = \rho q_m(t) - \beta_m q(t) \quad (25)$$

$$q_I(T) = q_m(T) = 0 \quad (26)$$

Now  $q_c(t)$  and  $q_I(t)$  are given, as before, by (13) and (14). We first solve for  $q_m(t)$ , assuming  $q(t) = q_I(t)$  and using (25) and (26). This gives,

$$q_m(t) = e^{-\rho(T-t)} \frac{\beta_I \beta_m}{\rho^2} \left[ \frac{\beta_c}{\rho} \{e^{\rho(T-t)} - 1\} - \{(\beta_c - \rho)\rho \left(\frac{T^2}{2} - tT + \frac{t^2}{2}\right) + \beta_c(T-t)\} \right] \quad (27)$$

From (14) and (27)

$$q_I(T-t) - q_m(T-t) = \left[ \frac{\beta_m}{2}(\beta_c - \rho)x + \rho^2 + \beta_c(\beta_m - \rho) - \frac{(e^x - 1)}{x} \beta_c(\beta_m - \rho) \right] x e^{-x} \frac{\beta_I}{\rho^3} \quad (28)$$

where  $x = \rho t$ .

If we call the bracketed expression on the R.H.S. of (28)  $g(x)$ , then it is easy to see that there exists a unique positive  $x^{**}$  such that  $g(x^{**}) = 0$ .

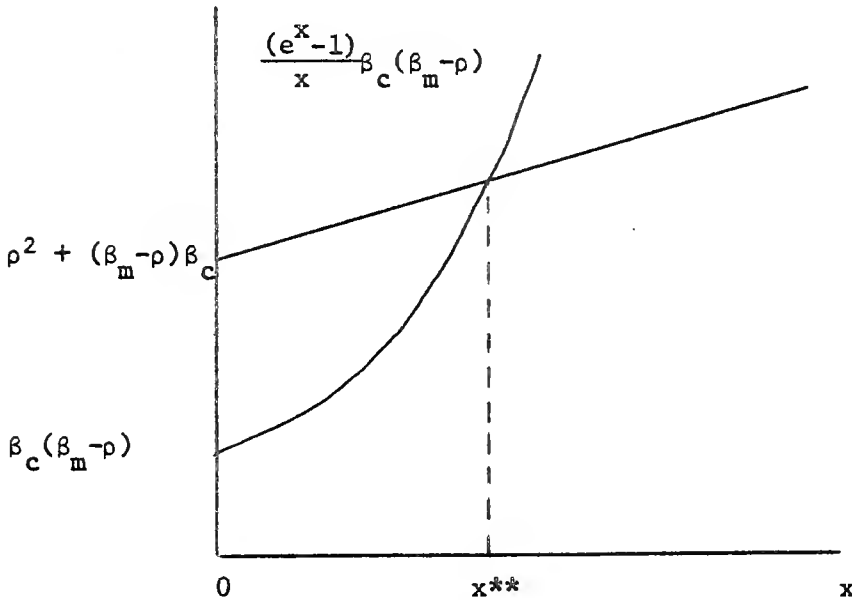


FIGURE 3

This is shown in Figure 3 above. It is also easy to see that  $g'(x^{**}) < 0$ .

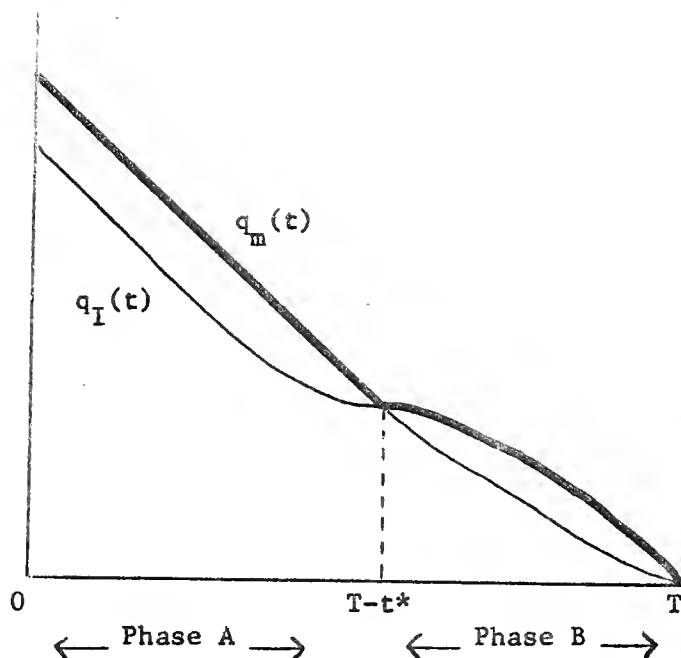


FIGURE 4

All this means that there exists a unique positive  $t^*$  such that

$$q_I(T-t) - q_m(T-t) \begin{cases} \geq 0 & \text{as } t \leq t^* \\ < 0 & \text{as } t > t^* \end{cases} \quad (29)$$

Figure 4 depicts this. From (21) we may now say that on the optimum path from the beginning up to time  $T-t^*$  all of the flow of machine tools currently available is used in producing more machine tools (i.e.,  $\lambda = 0$ ) and after that all of it is used in producing tractors (i.e.,  $\lambda = 1$ ). Let us call the first phase (when  $\lambda = 0$ ) A and the second phase (when  $\lambda = 1$ ) B. We have yet to find out the pattern of  $\alpha_i$ 's (indicating the allocation of foreign exchange) in these two phases.

As in Section II,  $q_c(t)$  is always greater than unity until  $q_c(T) = 1$ . From (24) this means, once again, that  $\alpha_c$  is always zero, i.e., on the optimum path we spend no part of our foreign exchange in importing consumer goods.

For finding out the pattern of  $\alpha_I(t)$  we have to compare  $q_c(t)$  with  $q(t)$ . From (22) we know that the path of  $q(t)$  is depicted by the heavy line



in Figure 4. Both  $q(t)$  and  $q_c(t)$  are declining functions of time.

From (9), (10) and (25) it is obvious that  $q(t)$  and  $q_c(t)$  curves cannot cross more than once in either Phase A or Phase B. In fact, one can prove a stronger result: in both the phases taken together there cannot be more than one crossing of the  $q(t)$  and  $q_c(t)$  curves. Let us give a short proof of the latter proposition.

Suppose  $q(t)$  and  $q_c(t)$  curves cross twice, once at time  $T-\tau'$  ( $t^* < \tau'$ ) and again at time  $T-\tau$  ( $t^* > \tau$ ), so that  $q_c(T-\tau') = q(T-\tau') = q_m(T-\tau')$  and  $q_c(T-\tau) = q(T-\tau) = q_I(T-\tau)$ . With  $q_c(T) = 1$  and  $q(T) = 0$ , this implies that

$$\dot{q}_c(T-\tau) > \dot{q}_I(T-\tau) \quad (30)$$

$$\text{and} \quad \dot{q}_c(T-\tau') < \dot{q}_m(T-\tau') \quad (31)$$

From (9) and (10), (30) implies

$$q_c(T-\tau) = q_I(T-\tau) > \frac{\beta_c}{\beta_I} \quad (32)$$

From (9) and (25), (31) implies

$$\frac{\beta_c}{\beta_m} > q_c(T-\tau') = q_m(T-\tau') \quad (33)$$

We know from (29) that

$$q_I(T-t^*) = q_m(T-t^*)$$

$$\text{and} \quad \dot{q}_I(T-t^*) > \dot{q}_m(T-t^*)$$

From (10) and (25) this means that

$$q_m(T-t^*) = q_I(T-t^*) > \frac{\beta_I}{\beta_m} q_c(T-t^*) \quad (34)$$

Now if we put (32), (33) and (34) together,

$$\frac{\beta_c}{\beta_m} > q_m(T-\tau') > q_m(T-t^*) > \frac{\beta_I}{\beta_m} q_c(T-t^*) > \frac{\beta_I}{\beta_m} q_c(T-\tau) > \frac{\beta_c}{\beta_m},$$

which is a contradiction. This means that  $q(t)$  and  $q_c(t)$  curves can never cross more than once. In other words, there is no "reswitching."

Let us now prove that if  $\beta_I \geq \beta_m$ ,  $q_c(t)$  curve must cross the  $q(t)$  curve in what we have called Phase B, as in our Figure 5. If the  $q_c(t)$  curve does not cross the  $q(t)$  curve in Phase B, since  $q_c(T) = 1$  and since in Phase B  $q(t) = q_I(t)$ , this means

$$q_c(T-t^*) > q_I(T-t^*)$$

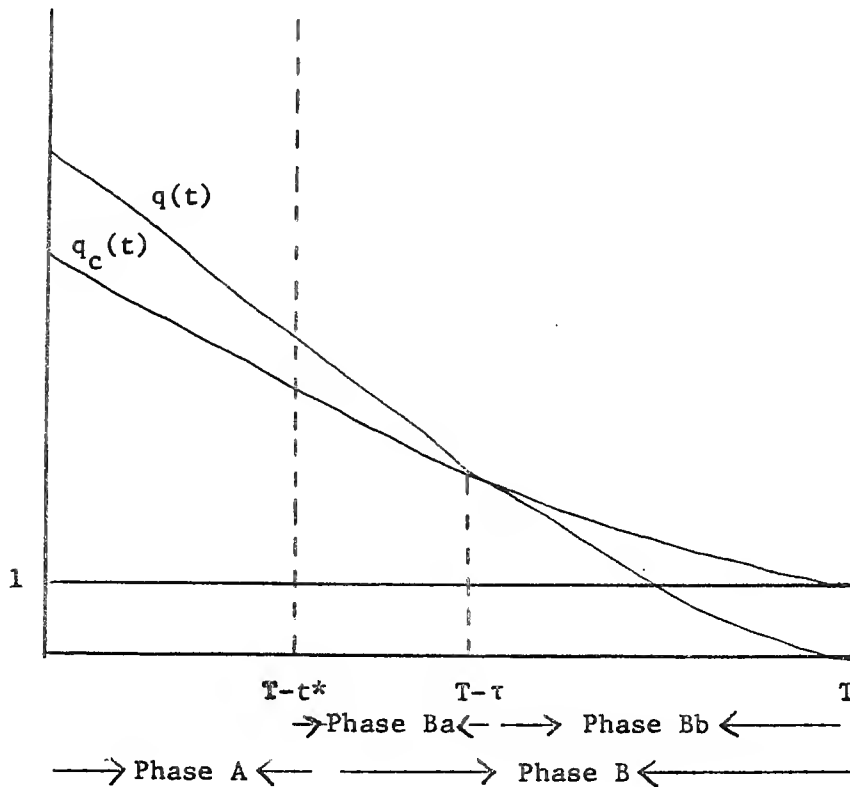


FIGURE 5

If  $\beta_I \geq \beta_m$  this is in contradiction with (34).

So in Figure 5, Phase B is in two sub-phases, Phase Ba ( $T - \tau > t > T - \tau^*$ ) when  $q(t) > q_c(T)$  and hence  $\alpha_I = 0$  along with  $\lambda = 1$ , and Phase Bb ( $T > t > T - \tau$ ) when  $q_c(t) > q(t)$  and hence  $\alpha_I = 1$  along with  $\lambda = 1$ . Since  $q_c(t)$  and  $q(t)$  curves can cross only once, in Phase A  $q(t)$  curve lies always about  $q_c(t)$  curve and  $\alpha_I = 0$  along with  $\lambda = 0$ .

What if  $\beta_m > \beta_I$ ? In this case we are not so sure. We, of course, know that the  $q_c(t)$  and  $q(t)$  curves can cross at most once. We can go somewhat beyond and say that if the planning horizon,  $T$ , is sufficiently large, there must be one crossing. This is because the expression for  $[q_m(0) - q_c(0)]$  is positive for  $T$  large, and with  $q_m(0) > q_c(0)$  there must be a crossing of  $q_c(t)$  and  $q(t)$ , either in Phase A or in Phase B. So for large  $T$  the  $q_c(t)$  curve cannot lie always above  $q(t)$  curve to make  $\alpha_I$  always equal to unity.

In the summary of this section, if  $\beta_I \geq \beta_m$ , i.e., if the output-capital ratio in the tractors-sector is not less than that in the machine-tools sector, or roughly speaking, if machine tools are not less "capital-intensive" than tractors, the optimum path has three uniquely determined phases: the first phase when we spend all of our foreign exchange in importing machine tools and devote all of our currently available flow of machine tools to the production of more machine tools; the second phase when we spend all of our foreign exchange in importing machine tools but use all machine tools for producing tractors; and the third phase when we spend all of our foreign exchange in importing tractors and use all of the current output of machine tools in producing tractors. The length of the last two phases is independent of  $T$ , the planning horizon, and the longer is the horizon the more the first phase predominates.

When machine tools are less capital-intensive than tractors, then also there are two distinct phases, one phase of all machine tools used in producing more machine tools, followed by the next phase when all machine tools are used in producing tractors. Again, the longer the planning horizon, the more the first phase will predominate. About the allocation of foreign exchange there is less certainty, but either we spend all of our foreign exchange in importing tractors or, if the planning horizon is long enough, we spend all of our foreign exchange first in importing machine tools and then spend all of it in importing tractors; in no case there is import of consumer goods and in no case there is subsequent reswitching to an earlier allocation policy.

We have so far assumed that ours is a small country which takes world prices as given and receives at each point of time a constant amount of foreign exchange through foreign aid. Let us now change the model and assume that we earn this foreign exchange through exports of our consumer goods, which we shall call  $X_c$ . Suppose we are an important exporter of these consumer goods and can therefore affect their world price,  $P_c$ , but when we buy imports (of tractors and machine tools) we are not big enough buyers in the world market for these to affect their prices so that the latter are again taken as given. This is not an unfamiliar situation for many developing countries.

For simplification we take a specific world demand function for our exports of consumer goods:

$$X_c(t) = P_c^\eta(t) \text{ with } \eta < 0, 1 + \eta < 0$$

where  $\eta$  is the constant price elasticity of demand, the absolute value of which is assumed greater than unity. So our earning of foreign exchange at time  $t$  is now given by our export receipts

$$P_c(t) \cdot X_c(t) = X_c^{\frac{1+\eta}{\eta}}(t) \quad (35)$$

The Hamiltonian H in (21) has now to be rewritten as

$$\begin{aligned} He^{\rho t} = & [\beta_c K_c(t) - X_c(t)] + q_c(t) [\beta_I K_I(t) + \alpha_I(t) \cdot X_c^{\frac{1+\eta}{\eta}}(t)] \\ & + q(t) [\beta_m K_m(t) + \{1 - \alpha_I(t)\} X_c^{\frac{1+\eta}{\eta}}(t)] \end{aligned} \quad (36)$$

We assume positive consumption and positive exports so that

$$\beta_c K_c(t) > X_c(t) > 0$$

If we now maximize the Hamiltonian with respect of  $\alpha_I(t)$ ,  $\lambda(t)$  and  $X_c(t)$ , we get (21), (22) and (23) as before (with  $\alpha_c=0$ ) and the extra condition

$$[\alpha_I(t) q_c(t) + \{1 - \alpha_I(t)\} q(t)] = \frac{\eta}{1+\eta} X_c^{\frac{1}{\eta}}(t) \quad (37)$$

(9), (10), (11), (25) and (26) will also stand as before.

So the analysis of the behaviour of  $q_c(t)$ ,  $q_I(t)$  and  $q_m(t)$  curves - and hence the behaviour of  $\alpha_I(t)$ ,  $\alpha_m(t)$  and  $\lambda(t)$  - is exactly the same. The only new condition is (37) which implies that we should choose the quantity of our exports in such a way that the marginal terms of trade (which is less than the average terms of trade)  $P_c(t) [\frac{1+\eta}{\eta}]$  should be equal to the reciprocal of  $\bar{q}(t)$ , where  $\bar{q}(t) = \max[q_c(t), q(t)]$ . Since both  $q_c(t)$  and  $q(t)$  are declining functions of time, it is easy to see that on the optimum path we export less and less over time and retain more for consumption.

Going back to the case when we have no exports but get a constant amount of foreign exchange through foreign aid, in the Appendix we consider the same allocation problem as in this section, but the planning horizon is infinite and

the maximand is a discounted (strictly) concave utility functional. This seems to be a fairly difficult problem (in the sense that it is hard to keep track of all the possible switchings of the  $q_1(t)$  curves). All we can report here on the basis of our calculations in the Appendix is as follows: (a) In the terminal phase of the optimal trajectory the available flow of machine tools is used in producing both tractors and machine tools (i.e.,  $1 > \lambda > 0$ ); (b) In the terminal phase of the optimal path we spend all of our foreign exchange in importing machine tools if  $\beta_I > \beta_m$  and  $\beta_c \geq \beta_m$ , or all of it in importing tractors if  $\beta_m > \beta_I$  and  $\beta_m > \beta_c$ , or all of it in importing consumer goods if  $\beta_m > \beta_I$  and  $\beta_c > \beta_m$ .

There are at least two important ways our model in this paper should be extended, although we have not done it here. One is to introduce intermediate products which are produced and imported and are used in other sectors. The other is to introduce labour as a primary factor of production.<sup>6/</sup> Both of these will make the analysis of our present problem much more complicated. But one can be sure that the introduction of a second factor of production (like labour) with diminishing marginal productivity of factors will add sufficient concavity to the maximand to do away with the extreme "bang-bang" type allocation policy in much of this paper.

Quite often a rough justification for not using labour as a factor of production in simple development planning models is cited in terms of the "surplus labour" hypothesis for many over-populated developing economies. Even if one overlooks the empirical doubts about this hypothesis, this ignores the institutional factors under which any transfer of labour from the food-grains-producing sector to the machine-producing sector raises the problem of

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6. For an analysis of optimum growth in a two-sector closed-economy non-shiftable capital model with labour as a shiftable primary factor of production, see Bose [3] and Ryder [14].

getting enough grains to feed the additional labourers. (This problem is analyzed in the context of an open economy by the author elsewhere [1].) From the standpoint of development planning under the typical institutional constraints food is a kind of capital good (this is in some sense akin to the concept of wages-fund or circulating capital of the classical economists) and in a more complete model of optimum foreign exchange allocation than the one studied in the present paper, this increases the importance of spending foreign exchange on food.

# APPENDIX

Suppose the social objective is to maximize the integral of a constant-elasticity instantaneous utility function over an infinite horizon:

$$\text{Max} \int_0^{\infty} \frac{C(t)^{1-\theta}}{1-\theta} e^{-\rho t} dt.$$

Subject to (2), (3), (4) and (5), where  $\theta$  is the absolute value of the elasticity of marginal utility.

The necessary conditions for optimality are given by (7), (10), (38) and (39):

$$\alpha_c(t) \begin{cases} = 0 & \text{if } \max[q_c(t), q_I(t)] > C(t) \\ \in [0, 1] & \text{if } \max[q_c(t), q_I(t)] = C(t) \\ = 1 & \text{if } \max[q_c(t), q_I(t)] < C(t) \end{cases} \quad (38)$$

$$\dot{q}_c(t) = \rho q_c(t) - \beta_c C(t)^{-\theta} \quad (39)$$

Since the utility function is concave the following transversality conditions are sufficient for optimality:

$$\lim_{t \rightarrow \infty} q_i(t) e^{-\rho t} K_i(t) = 0; \quad \lim_{t \rightarrow \infty} q_i(t) \geq 0, \quad i=c, I. \quad (40)$$

We shall prove here that the asymptotically optimal policy in this case is to spend all of foreign exchange in importing machine tools (i.e.,  $\alpha_m=1$ ,  $\alpha_c=\alpha_I=0$ ).

From (39) and (10)

$$q_c(t) = e^{\rho t} [q_c(0) - \beta_c \int_0^t e^{-\rho v} \psi(v) dv] \quad (41)$$

$$q_I(t) = e^{\rho t} [q_I(0) - \beta_I \int_0^t e^{-\rho v} q_c(v) dv] \quad (42)$$

where  $\psi(t) \equiv C(t)^{-\theta}$



Since  $K_c(t)$  and  $K_I(t)$  are increasing functions of time, the transversality conditions (40) imply that

$$q_c(0) = \beta_c \int_0^{\infty} e^{-\rho v} \psi(v) dv \quad (43)$$

$$\text{and} \quad q_I(0) = \beta_I \int_0^{\infty} e^{-\rho v} q_c(v) dv \quad (44)$$

Using (43) and (44) in (41) and (42),

$$q_c(t) = e^{\rho t} \beta_c \int_t^{\infty} e^{-\rho v} \psi(v) dv \quad (45)$$

$$q_I(t) = e^{\rho t} \beta_I \int_t^{\infty} e^{-\rho v} q_c(v) dv \quad (46)$$

We shall now prove that if  $\alpha_c = \alpha_I = 0$ ,  $q_I(t) > q_c(t)$  and  $q_I(t) > \psi(t)$  as  $t \rightarrow \infty$ .

From (45) and (46),

$$q_I(t) - q_c(t) = e^{\rho t} \int_t^{\infty} e^{-\rho v} [\beta_I q_c(v) - \beta_c \psi(v)] dv \quad (47)$$

Call  $V(t) = \beta_I q_c(t) - \beta_c \psi(t)$ . Then from (45),

$$\begin{aligned} V(t) &= \beta_I e^{\rho t} \beta_c \int_t^{\infty} e^{-\rho v} \psi(v) dv - \beta_c \psi(t) \\ &= \frac{\beta_I}{\rho} e^{\rho t} \beta_c [e^{-\rho t} \psi(t) + \int_t^{\infty} e^{-\rho v} \psi'(v) dv - \lim_{t \rightarrow \infty} e^{-\rho v} \psi(v)] \\ &\quad - \beta_c \psi(t) \end{aligned}$$

through integration by parts,

$$= \beta_c \psi(t) \frac{(\beta_I - \rho)}{\rho} + \frac{\beta_I \beta_c}{\rho} e^{\rho t} \int_t^{\infty} e^{-\rho v} \psi'(v) dv$$

since  $\lim_{v \rightarrow \infty} e^{-\rho v} \psi(v) = 0$ .

Therefore  $V(t) > 0$ , if

$$\frac{(\beta_I - \rho)}{\beta_I} > \frac{- \int_t^{\infty} e^{-\rho v} \psi'(v) dv}{\psi(t) e^{-\rho t}} \quad (48)$$

Let us call the R.H.S. of (48)  $\frac{f(t)}{g(t)}$ . We know that  $f(\infty) = 0$  and  $g(\infty) = 0$ .

So

$$\lim_{t \rightarrow \infty} \frac{f(t)}{g(t)} = \lim_{t \rightarrow \infty} \frac{f'(t)}{g'(t)} = \lim_{t \rightarrow \infty} \frac{1}{[1 - \rho \frac{\psi(t)}{\psi'(t)}]} = \lim_{t \rightarrow \infty} \frac{1}{[1 + \frac{\rho K_c(t)}{\theta K_c(t)}]} = 0$$

since  $\psi(t) = C(t)$ ,  $C(t) = \beta_c K_c(t)$ , and with  $\alpha_I = 0$  and  $\alpha_m = 1$ ,

$$\frac{\dot{K}_c(t)}{K_c(t)} \rightarrow 0 \text{ as } t \rightarrow \infty.$$

So from (48), as  $t \rightarrow \infty$ ,  $V(t) > 0$  and hence (47) is positive. Now from

(46)

$$\begin{aligned} q_I(t) - \psi(t) &= e^{\rho t} \beta_I \int_t^{\infty} e^{-\rho v} q_c(v) dv - \psi(t) \\ &= e^{\rho t} \beta_I \int_t^{\infty} \int_v^{\infty} e^{-\rho \tau} \beta_c \psi(\tau) d\tau dv - \psi(t) \\ &= e^{\rho t} \frac{\beta_I \beta_c}{\rho} \int_t^{\infty} e^{-\rho v} \psi(v) [1 - X(v)] dv - \psi(t) \\ &\quad - \int_t^{\infty} e^{-\rho \tau} \psi'(\tau) d\tau \end{aligned} \quad (49)$$

$$\text{where } X(v) = \frac{v}{e^{-\rho v} \psi(v)}.$$

We have already proved that  $X(v) \rightarrow 0$ , as  $v \rightarrow \infty$ . (49) will be positive if we can prove that as  $t \rightarrow \infty$

$$\frac{\int_t^{\infty} e^{-\rho v} \psi(v) [1 - X(v)] dv}{e^{-\rho t} \psi(t)} > \frac{\rho}{\beta_I \beta_c} \quad (50)$$

Let us call the R.H.S. of (50),  $\frac{s(t)}{p(t)}$ .

We know that  $s(\infty) = 0$  and  $p(\infty) = 0$ . So

$$\lim_{t \rightarrow \infty} \frac{s(t)}{p(t)} = \lim_{t \rightarrow \infty} \frac{s'(t)}{p'(t)} = \lim_{t \rightarrow \infty} \frac{[1 - X(t)]}{[\rho - \frac{\psi'(t)}{\psi(t)}]} = \lim_{t \rightarrow \infty} \frac{[1 - X(t)]}{[\rho + \frac{K_c(t)}{K_c(t)}]} = \frac{1}{\rho} \quad (51)$$

Since  $\beta_I > \rho$ ,  $\beta_c > \rho$ , (51) implies the validity of (50). Thus we have proved that as  $t \rightarrow \infty$ ,  $q_I(t) > q_c(t)$  and  $q_I(t) > \psi(t)$  and hence  $\alpha_c = \alpha_I = 0$  and  $\alpha_m = 1$  is the asymptotically optimal policy.

Let us now take the allocation problem of Section III, when we have domestic production of machine tools, but introduce an infinite horizon and a non-linear instantaneous utility function of the form  $U(c) = \frac{C(t)}{1-\theta}$ . Let us maximize the utility functional subject to (2), (3), (5), (17), (18) and (19). The necessary conditions for optimality are given by (21), (22), (23), (38), (10), (25) and (39). The transversality conditions are given by (40). Now if the system remains in the phase when  $1 > \lambda(t) > 0$  for an interval  $[t_0, t_1]$ , then for  $t_1 > t > t_0$ :

$$q_I(t) = q_m(t) = q^*(t) \quad (52)$$

Using (39), (10) and (25), (52) implies that

$$q^*(t) = \frac{\beta_I}{\beta_m} q_c(t) = \frac{\beta_I \beta_c}{\beta_m^2} \psi(t) \quad (53)$$

Using (22) and (38), (53) means that for  $t_1 > t > t_0$ , if  $\beta_I > \beta_m$  and  $\beta_c \geq \beta_m$ , then  $q^*(t) > q_c(t)$  and  $q^*(t) > \psi(t)$ , so that

$$\alpha_I = \alpha_c = 0 \text{ and } \alpha_m = 1; \quad (54)$$

if  $\beta_m > \beta_I$  and  $\beta_m > \beta_c$ , then  $q^*(t) < q_c(t)$  and  $q_c(t) > \psi(t)$ , so that

$$\alpha_c = \alpha_m = 0 \text{ and } \alpha_I = 1; \quad (55)$$

and if  $\beta_m > \beta_I$  and  $\beta_c > \beta_m$ , then  $q^*(t) < q_c(t)$  and  $q_c(t) < \psi(t)$ , so that

$$\alpha_I = \alpha_m = 0 \text{ and } \alpha_c = 1. \quad (56)$$

Let us first take the case when the sectoral capital-intensities are such that (54) is valid. In this case

$$K_c(t) = K_c(t_0) e^{\frac{(\beta_m - \rho)}{\theta}(t - t_0)} \quad (57)$$

$$K_I(t) = \frac{(\beta_m - \rho)}{\theta \beta_I} K_c(t) \quad (58)$$

$$\begin{aligned} \text{and } K_m(t) = [K_m(t_0) + \frac{F}{\beta_m} - \frac{K_c(t_0)(\beta_m - \rho)^2}{\theta \beta_I \{\rho - \beta_m(1 - \theta)\}}] e^{\beta_m(t - t_0)} \\ - \frac{F}{\beta_m} + \frac{K_c(t)(\beta_m - \rho)^2}{\theta \beta_I \{\rho - \beta_m(1 - \theta)\}} \end{aligned} \quad (59)$$

We assume that  $\{\rho - \beta_m(1 - \theta)\}$  is positive which is a boundedness assumption on the maximand integral of the problem. From (53), (57), (58) and (59) it follows that

$$\lim_{t \rightarrow \infty} e^{-\rho t} q_c(t) K_c(t) = \lim_{t \rightarrow \infty} \frac{\{\beta_c K_c(t_0)\}^{1-\theta}}{\beta_m} e^{-\{\rho - (1-\theta)\beta_m\} \frac{(t-t_0)}{\theta}} = 0 \quad (60)$$

$$\lim_{t \rightarrow \infty} e^{-\rho t} q_I(t) K_I(t) = \lim_{t \rightarrow \infty} \frac{\beta_c^{1-\theta} (\beta_m - \rho) K_c^{1-\theta}(t_0)}{\beta_m^2 \theta} e^{-\{\rho - (1-\theta)\beta_m\} \frac{(t-t_0)}{\theta}} = 0 \quad (61)$$

but

$$\begin{aligned} \lim_{t \rightarrow \infty} e^{-\rho t} q_m(t) K_m(t) = 0, \text{ only if} \\ K_m(t_0) + \frac{F}{\beta_m} = \frac{K_c(t_0)(\beta_m - \rho)^2}{\theta \beta_I \{\rho - \beta_m(1 - \theta)\}} \end{aligned} \quad (62)$$

so that

$$K_m(t) = \frac{K_c(t)(\beta_m - \rho)^2}{\theta^2 \beta_I \{\rho - \beta_m(1 - \theta)\}} - \frac{F}{\beta_m} \quad (63)$$

So in the phase when  $1 > \lambda(t) > 0$  the only trajectory that satisfies the transversality conditions is the one characterized by solutions of  $K_c(t)$ ,  $K_I(t)$  and  $K_m(t)$  as given by (56), (57) and (63). As  $t \rightarrow \infty$  this path also dominates any other path with  $\lambda = 0$  or  $\lambda = 1$ , so that this must be the terminal phase of the optimal trajectory.  $\lambda^*$ , the proportion of currently available machine tools to be used to producing tractors on this terminal phase of the optimal trajectory is given by:

$$\lambda^* = \frac{\rho - \beta_m(1-\theta)}{\theta\beta_m} \in [0,1] \quad (64)$$

Using similar methods we can show that if the sectoral capital-intensities are such that (55) is valid, then the transversality conditions are satisfied in the phase when  $1 > \lambda(t) > 0$  only if

$$K_m(t_0) = \frac{K_c(t_0)(\beta_m - \rho)^2}{\theta\beta_I\{\rho - \beta_m(1-\theta)\}} \quad (65)$$

Once again this must be the terminal phase of the optimal trajectory and  $\lambda^*$  is the same as in (64). Similarly, we can show that if the sectoral capital-intensities are such that (56) is valid, then the transversality conditions are satisfied in the phase when  $1 > \lambda(t) > 0$  only if

$$K_m(t_0) = \frac{(\beta_m - \rho)^2}{\theta\beta_I} \left[ \frac{F}{\theta\beta_c\beta_m} + \frac{K_c(t_0)}{\{\rho - \beta_m(1-\theta)\}} \right] \quad (66)$$

Again, this must be the terminal phase of the optimal trajectory.  $\lambda^*(t)$  in this case is given by

$$\lambda^*(t) = \frac{\beta_c K_c(t) + F}{\theta\beta_c\beta_m K_c(t) + F + \frac{\rho - \beta_m(1-\theta)}{\theta\beta_m}} \in [0,1] \quad (67)$$

as  $t \rightarrow \infty$ ,  $\lambda^* \rightarrow \frac{\rho - \beta_m(1-\theta)}{\theta\beta_m}$ .

## REFERENCES

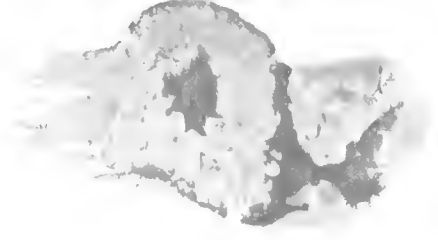
- [1] P.K. Bardhan, Economic Growth, Development and Foreign Trade: A Study in Pure Theory, John Wiley & Sons, (Interscience Division), forthcoming, Ch. 9.
- [2] S. Bose, "Optimal Growth and Investment Allocation," Review of Economic Studies, October 1968.
- [3] S. Bose, "Optimal Growth in a Non-Shiftable Capital Model," unpublished.
- [4] S. Chakravarty, "Optimal Programmes of Capital Accumulation in a Two-Sector Model with Non-Shiftable Capital," Working Paper No. 17A, Delhi School of Economics, January 1967.
- [5] P. Dasgupta, "Optimum Growth When Capital is Non-Transferable," Review of Economic Studies, January, 1969, forthcoming.
- [6] M. Dobb, An Essay on Economic Growth and Planning, London, 1960.
- [7] E.D. Domar, "A Soviet Model of Growth," Essays in the Theory of Economic Growth, Oxford University Press, New York, 1967.
- [8] G.A. Fel'dman, "On the Theory of Growth Rates of National Income," translated in N. Spulber (ed.), Foundations of Soviet Strategy for Economic Growth, Indiana University Press, Bloomington, 1964.
- [9] L. Johansen, "Some Theoretical Properties of a Two-Sector Model of Optimal Growth," Review of Economic Studies, January 1967.
- [10] P.C. Mohalanobis, "Some Observations on the Process of Growth of National Income," Sankhya, September 1953.
- [11] P.C. Mohalanobis, "Science and National Planning," Sankhya, September 1958.
- [12] K.N. Raj, "Role of the Machine-Tools Sector in Economic Growth," in C.H. Feinstein (ed.), Socialism, Capitalism and Economic Growth, Essays presented to Maurice Dobb, Cambridge University Press, 1967.
- [13] K.N. Raj and A.K. Sen, "Alternative Patterns of Growth under Conditions of Stagnant Export Earnings," Oxford Economic Papers, February 1961.
- [14] H.E. Ryder, "Optimal Accumulation in a Two-Sector Neo-Classical Economy With Non-Shiftable Capital," Journal of Political Economy, forthcoming.
- [15] R.M. Solow, "Some Problems in the Theory and Practice of Economic Planning," Economic Development and Cultural Change, January 1962.











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